The Enhanced Gain Extrapolation Technique

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***Abstract*—We present an overview of a recently developed technique for performing antenna gain measurements with gain extrapolation that uses significantly fewer data points and at shorter distances than traditional gain extrapolation. This enhanced technique purposely incorporates third-order mutual coupling between antennas, which can be thought of as a useful homodyne signal, rather than an unwanted degradation of the antenna-to-antenna coupling signal as has been the historically accepted viewpoint. From Wacker's fundamental extrapolation equations, we give the development of the third-order signal, which underpins this technique. From the third-order signal, the framing of gain extrapolation can be approached as a measure of interference fringes, as opposed to a by-rote curve fitting problem, and thus provides ways of specifying the number of required data points and measurement distances so as to reduce both significantly from the traditional gain extrapolation approach. The truncation order of the full signal expansion, as it relates to the conditioning of the problem, is presented in light of the behavior of the design matrix that defines the gain extrapolation scenario and the orders of scattering, thus leading to fewer required samples. Along with considerations of the matrix conditioning, guidelines are presented from the third-order signal and interference fringes for sampling criteria and sampling accuracy criteria. These guidelines aid in choices of measurement system accuracy and precision requirements based on known values of the operating frequency, wavelength, and antenna dimensions. Bounds for gain uncertainty based on these sampling criteria are also given. Results comparing NIST reference antenna measurements made with the traditional gain extrapolation and enhanced gain extrapolation technique are presented. It is shown that the enhanced technique can produce gain values in agreement to within uncertainties of the traditional technique for the reference antennas.**

***Index Terms*—Antenna, antenna gain, gain, gain extrapolation.**

I. Introduction

Traditional gain extrapolation [1]-[4], which has been around since the 1970's, makes use of Wacker's series expansion [5] for the transmitted signal $S_{ij}d$ including all orders of scattering between two antennas "*i*" and "*j*" at a separation distance d. This series expansion is shown in (1) and (2).

$$S_{ij}d = \sum_{p=0}^{\infty} \frac{e^{i(2p+1)kd}}{d^{2p}} \sum_{q=0}^{\infty} \frac{A_{pq}}{d^q} \tag{1}$$

$$\begin{aligned} S_{ij}d = e^{ikd}(A_{00} + \frac{A_{01}}{d} + \frac{A_{02}}{d^2} + \frac{A_{03}}{d^3} + ...)+ \\ e^{ik3d}(\frac{A_{10}}{d^2} + \frac{A_{11}}{d^3} + \frac{A_{12}}{d^4} + \frac{A_{13}}{d^5} + ...)+ \\ e^{ik5d}(\frac{A_{20}}{d^4} + \frac{A_{21}}{d^5} + \frac{A_{22}}{d^6} + \frac{A_{23}}{d^7} + ...) + ... \ . \end{aligned} \tag{2}$$

In traditional gain extrapolation, the lowest order term, A_{00}, which represents the product of the antenna gains $g_i g_j$ (i.e. "pair-gain") of the two antennas, is isolated from the rest of the signal, where,

$$|A_{00}|^2 = \frac{g_i g_j}{4k^2} \ . \tag{3}$$

This is accomplished by means of averaging the oscillations observed in $S_{ij}d$ due to the higher orders of scattering, $e^{i(2p+1)kd}$ for $p > 1$, and keeping only the direct signal e^{ikd} represented by the series,

$$S_{ij}d = e^{ikd}(A_{00} + \frac{A_{01}}{d} + \frac{A_{02}}{d^2} + \frac{A_{03}}{d^3} + ...). \tag{4}$$

The pair-gain term, A_{00}, is determined by fitting a curve to the the square magnitude of (4). For gain standard horn antennas, for which gain extrapolation provides the highest accuracy calibration, $S_{ij}d$ is dominated by the third order scatter wave e^{i3kd} producing oscillations with a period of $\lambda/2$. To achieve effective averaging, $S_{ij}d$ must be measured at many separation distances spaced by at least $\lambda/10$, thus creating a dense data set, often times numbering in over 4000 data points per antenna pair for standard gain horn antennas.

In essence, the traditional approach just described is motivated by the notion that the higher orders of scattering are an unnecessary artifact that get in the way of observing the direct signal and thus to be eliminated. Upon first glance at Wacker's expansion this appears reasonable. However, the cost of this interpretation is that many data points must be used to determine antenna gain. Upon further examination, as we discuss below, the need for so many data points is not uniquely governed by the design of the problem, and that the problem is in fact amenable to using significantly fewer data points, if we abandon the requirement to do away with the oscillations produced by the third order scattering, given by

the e^{i3kd} term. It is important to stress the scope of this work applies to scenarios where (1) is sufficient to describe $S_{ij}d$. That is, in a measurement setting where other factors, such as multipath effects are not dominant, and where antennas exhibit pronounce third order mutual coupling. This is the case typically encountered for standard gain horn antennas and aperture type antennas [6], which are of interest for calibration purposes.

II. Antenna Gain from Third Order Signal

In the enhanced gain extrapolation technique [7], along with the direct signal e^{ikd}, the third order e^{i3kd} scattering component is retained, rather than averaged out. As such, (2) is truncated to the third order and the square magnitude is taken to obtain the third order homodyne signal, $I^{3^{rd}}$. It can be shown [7] the result is,

$$I^{3^{rd}} = I_0 + I_1 + 2\sqrt{I_0 I_1} cos\left(\frac{4\pi d}{\lambda} + \Delta\phi\right) \ , \qquad (5)$$

where, $I_0 \equiv |\Psi_0|^2$, and $I_1 \equiv |\Psi_1|^2$, with,

$$\Psi_0 = \sum_{q=0}^{\infty} \frac{A_{0q}}{d^q} = |\Psi_0| e^{i\phi_0} \ , \qquad (6)$$

$$\Psi_1 = \sum_{q=0}^{\infty} \frac{A_{1q}}{d^{(2+q)}} = |\Psi_1| e^{i\phi_1} \ , \qquad (7)$$

where $I^{3^{rd}}$, I_0, I_1 are implicitly functions of d, and λ is the operating wavelength. From (5) we can approach the gain extrapolation measurement from an interferometric point of view – the interference of direct and third order waves. Where (5) is appreciated as an interferogram [7],[8],[9] comprised of a series of diminishing interference fringes separated by a distance,

$$\Delta d = \Delta l \left(\frac{\lambda}{4}\right), \qquad (8)$$

for a change in fringe order Δl. In this framework, the gain measurement can be accomplished through the measurement of the fringes directly (rather than them being averaged out). In particular, the value of the maximum, $I^{3^{rd}}_{max}$, and minimum, $I^{3^{rd}}_{min}$, of the fringes can be used to isolate the pair-gain term of interest $|A_{00}|^2$ (see (3)). From (5) it can be shown that,

$$\frac{I^{3^{rd}}_{max} + I^{3^{rd}}_{min}}{2} = I_0 + I_1 \equiv \bar{I} \ , \qquad (9)$$

with,

$$\bar{I} = \sum_{n=0}^{\infty} \frac{A^{'}_{0n}}{d^n} = A^{'}_{00} + \frac{A^{'}_{01}}{d} + \frac{A^{'}_{02}}{d^2} + \frac{A^{'}_{03}}{d^3} + \frac{A^{'}_{04}}{d^4} + ... \ , \qquad (10)$$

and $A^{'}_{00} = |A_{00}|^2$. The measurement of interference fringes are thus directly related to $|A_{00}|^2$.

For a finite set of fringes measured about their center distance, d_c, the measurements of $\bar{I}$ can be arranged as elements of a vector Y with corresponding design matrix $\mathbf{M}$ and coefficient vector X so that, $Y = \mathbf{M}X$ is,

$$\begin{bmatrix} \bar{I}_0 \\ \bar{I}_1 \\ \vdots \\ \bar{I}_m \end{bmatrix} = \begin{bmatrix} 1 & 1/d_{c,0} & 1/d^2_{c,0} & 1/d^3_{c,0} & \dots 1/d^n_{c,0} \\ 1 & 1/d_{c,1} & 1/d^2_{c,1} & 1/d^3_{c,1} & \dots 1/d^n_{c,1} \\ & & & \vdots & \\ 1 & 1/d_{c,m} & 1/d^2_{c,m} & 1/d^3_{c,m} & \dots 1/d^n_{c,m} \end{bmatrix} \begin{bmatrix} A^{'}_{00} \\ A^{'}_{01} \\ A^{'}_{02} \\ \vdots \\ A^{'}_{0n} \end{bmatrix} , \qquad (11)$$

and $X = \mathbf{M}^{-1}Y$, where $\mathbf{M}^{-1}$ is the pseudoinverse of $\mathbf{M}$. It is then a matter of solving for $A^{'}_{00}$. Given three antennas that form pairs with pair-gains, $g_i g_j$, $g_i g_k$, $g_j g_k$ the gains, G_i, G_j, G_k for each antenna can be solved for by first implementing (11) to obtain the pair-gains then the individual gains solved for in the usual way as is done for any three-antenna method [3],[4],[7].

III. Sampling

The matrix $\mathbf{M}$ is of the form of a Vandermonde matrix in variable d^{-n} and of *finite* order n. Thus in determining the pair-gain, rather than a large data set whose purpose is to filter out the third order scattering as in traditional gain extrapolation, the measurement of fringes provides a means for sampling (1) within what is sufficient as governed by the conditioning of $\mathbf{M}$. That is, the sampling is now commensurate with the lowest sufficient order, n, of $\mathbf{M}$. The lowest sufficient order of $\mathbf{M}$ can be determined from it's singular values as these represent the contribution of the proximity terms $A^{'}_{0n}$ in (10) to $\bar{I}$. The proximity terms will naturally truncate the series as their contribution to $\bar{I}$ effectively diminishes with increasing order. This truncation order turns out to be around $n = 4$ when the third order scattering is dominant [7]. Figure 1 shows the lowest singular value of $\mathbf{M}$ as the order n is increased. From the figure it is seen that they become increasing small and tend toward zero for $n > 4$. As the minimum number of measured

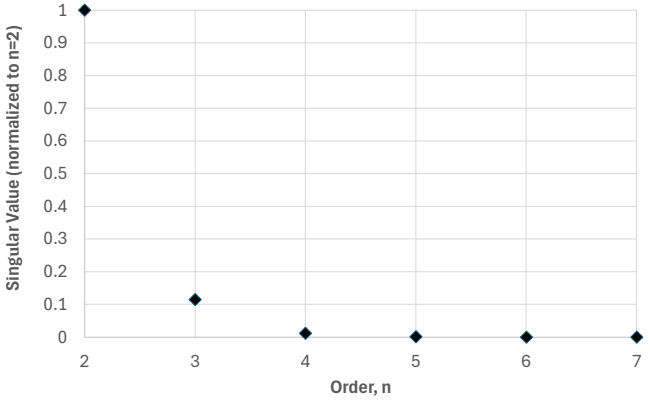


Fig. 1: Lowest singular value of $\mathbf{M}$ is shown as the order n is increased and are normalized for the $n = 2$ case. The singular values diminish rapidly toward zero for $n > 4$.

fringes needed to solve (11) is equal to $(n+1)$, the minimum

number of samples per antenna pair is thus $\mathbf{N} = 2(n+1)$. Where the factor of "2" accounts for two measurements per fringe (the max and min extrema). With $n = 4$, then $\mathbf{N} = 10$ which is significantly less than the ≈ 4000 data points needed per antenna pair in traditional gain extrapolation. Furthermore, from $\mathbf{M}$ it can be shown that conditioning is optimized when *samples are equally spaced* [7].

Sampling *accuracy* criteria can also be developed based on the visibility, $\mathcal{V}$, of the fringes [8],[9]. In essence, the max and min extrema, $I_{max}^{3^{rd}}$ and $I_{min}^{3^{rd}}$ of the fringes, prescribe specific measurement locations; the departure from which can cause errors in the values of $\bar{I}$. Furthermore, the sharper the fringes, the higher their visibility, and the more sensitive the error in $\bar{I}$ will be to the the accuracy of sampling position d at the extrema. These errors in $\bar{I}$ will manifest as an error in the pair-gain term $A_{00}^{'} = |A_{00}|^2$. This sampling accuracy criteria is a unique aspect of the enhanced technique as there is no such sampling accuracy criteria for the traditional gain extrapolation technique. The visibility in terms of the fringe extrema is given by [8],[9],

$$\mathcal{V} = \frac{I_{max}^{3rd} - I_{min}^{3rd}}{I_{max}^{3rd} + I_{min}^{3rd}} = \frac{2\sqrt{I_0 I_1}}{\bar{I}}, \tag{12}$$

where the visibility is bounded by the values $0 \leq \mathcal{V} \leq 1$. The error in the gain G due to a positional error δd under worst case conditions of $\mathcal{V} = 1$ is given by [7],

$$G_i^{'} - G_i = \delta G = \frac{10 Log_{10}(\alpha)}{2}\ , \tag{13}$$

where α is related to the change in G based on if δd occurs around the maximum or minimum extrema of a fringe and takes on the values,

$$\alpha = \begin{cases} \dfrac{1 + cos\,(2k\delta d)}{2}, & \text{Around Fringe Max} \quad (14) \\ \dfrac{3 - cos\,(2k\delta d)}{2}. & \text{Around Fringe Min} \quad (15) \end{cases}$$

Using (13),(14) and (15), the gain error due to sampling accuracy can be determined. For instance, for gain errors of $\delta G \approx \pm 0.01$ dB, the sampling accuracy would need to be $\delta d \approx \lambda/100$ and for an error of $\delta G \approx \pm 0.035$ dB the the sampling accuracy would be $\delta d \approx \lambda/50$.

In traditional gain extrapolation the farthest measurement separation distance is often taken as the "far-field" distance of $d_{Far} = 2D^2/\lambda$ where D is the largest dimension of the antenna apertures. However, in the enhanced technique this distance becomes compressed [7] since the third order wave travels three times the distance for a given separation distance, and thus becomes,

$$d_{Far} = \frac{2D^2}{3\lambda}\ . \tag{16}$$

The near distance is taken as the boundary of the Fresnel region about the antenna aperture so as to reduce perturbations of the fringes due to stronger high-order proximity coupling when antennas are too close,

$$d_{Near} = 0.62\sqrt{\frac{D^3}{\lambda}}\ . \tag{17}$$

For non-identical antennas, d_{Near} is the average of the Fresnel distances for the antenna apertures.

IV. Gain Measurements

Results for gain measurements of three Ku-band standard gain horn antennas using this enhanced technique are shown in Table I and associated uncertainty given in Table II. The values in the "Reference" column are those determined from the international CCEM comparison [10] of these antennas and include the results from traditional gain extrapolation technique, where approximately 3800 data points per antenna pair were used. The values in column "Measured" were determined using the enhanced technique just described where 10 data points per antenna pair were used. Measurements were performed on the Large Antenna Positioning System (LAPS) at the NIST Robotically Enhanced Antenna Lab for Metrology (REALM). Positional accuracy of the LAPS system was better than 20 μm, imparting a gain error based on (13) of $\delta G \approx \pm 0.003$. For further details on the uncertainty analysis using the enhanced technique the reader is referred to [7].

TABLE I: Ku-Band Antenna Gain Comparison

Antenna	Freq. (GHz)	Reference (dB)	Measured (dB)
SN3935	12.4	23.63 ± 0.1	23.62 ± 0.07
SN3935	15	24.44 ± 0.05	24.44 ± 0.07
SN3935	18	24.85 ± 0.06	24.80 ± 0.06
SN3936	12.4	23.62 ± 0.1	23.62 ± 0.07
SN3936	15	24.45 ± 0.08	24.46 ± 0.06
SN3936	18	24.83 ± 0.06	24.80 ± 0.07

V. Conclusion

We presented an overview of a newly developed enhanced gain extrapolation technique that capitalizes on antenna-to-antenna mutual coupling. This technique uses interference of the direct and third order scattered waves to obtain antenna gain with significantly fewer data points and at shorter distances than traditional gain extrapolation. The salient aspects of this technique were presented including matrix representation, conditioning of the problem, sampling criteria, and sampling-induced gain errors. Gain results were presented, for two NIST reference antennas operating in the WR-62 band, which agree to within uncertainties of the reference values.

TABLE II: Uncertainty

Sources of Uncertainty	**Standard Uncertainty (dB)**	
Polarization Mismatch	< 0.01	< 0.01
RF System Variability	0.01	0.01
Impedance Mismatch	0.01	0.01
Antenna Alignment	0.01	0.01
Matrix Solution	***0.01***	***0.02***
Positional Error	*0.003*	*0.003*
Connector Repeatability	0.01	0.01
Residual Multipath	0.02	0.02
Random Uncertainties	0.01	0.01
Expanded Uncertainty (dB) (k=2)	**0.06**	**0.07**

Note for Table II: The two columns show the uncertainty breakdown and that the difference between the the +/-0.07 dB and +/-0.06 dB uncertainties in Table I arises from the variation in Matrix Solution uncertainty contribution. See [7] for more details.

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